THE PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 131, NO. 4 — 15 AUGUST 1963

Magnetodynamic Instabilities in a Plasma-Beam System

JACOB NEUFELD AND HARVEL WRIGHT

Health Physics Division, Oak Ridge National Laboratory, Oak Ridge, Tennessee*

(Received 25 February 1963)

It is well known that Alfvén waves may be excited in a plasma-beam system. It is found in this investigation that such excitation is different for an ordinary (Alfvén-O) wave and for an extraordinary (Alfvén-X) wave. An Alfvén-O wave is characterized by a nonisotropic velocity distribution. However, the instability behavior for such a wave is found to be isotropic, i.e., the excited frequency and the rate of growth are independent of the direction of propagation of the wave. On the other hand, an Alfvén-X wave has an isotropic velocity distribution, but its instability behavior is found to be nonisotropic. Expressions are derived for the excited frequency and the rate of growth corresponding to any direction of an Alfvén-O or Alfvén-X wave. An analysis is made of an excited Alfvén wave aligned along the direction of the magnetic field and a relationship given between the instabilities of linearly and circularly polarized waves. A comparison is made of instabilities produced by an electron and by an ion beam.

1. INTRODUCTION

THIS investigation deals with a charge equilibrated system comprised of a beam of ions or electrons interacting with a stationary plasma and aligned in the direction of an externally applied magnetic field $\mathbf{B}_0$. Under certain conditions such an interaction results in a magnetodynamic instability,[1] in which the kinetic energy of the beam is gradually converted into the energy of a growing magnetodynamic wave.

According to a recent investigation,[2] (designated as Paper 1), a magnetodynamic instability can be produced in a plasma by a beam of protons or electrons for various velocities of such a beam. Of particular interest in this connection are "dense" plasmas[3] such as the ionosphere, interstellar clouds, thermonuclear discharges, etc. In such plasmas, a growing magnetodynamic wave occurs for a very extended velocity range of the incident proton beam. On the other hand, if the incident beam consists of electrons, the velocity of the beam must be much higher than the velocity of a proton beam necessary for excitation of a similar wave. In any plasma, other than a very dense plasma, an electron beam must have a velocity in the ultrarelativistic range in order to excite a magnetodynamic wave.

It is known that an electromagnetic wave may be excited by a beam of small intensity only in case of resonance between the wave and the beam.[2,4] Such a resonance condition is the basis of the method used in this investigation. In recent investigations by Dokuchaev,[5] Stepanov and Kitzenko,[6] Kovner,[7] and by Akhiezer, Kitzenko, and Stepanov,[8] the resonance requirement was not taken into account. Therefore, the results obtained here are different from the results obtained in these other investigations.

* Operated by Union Carbide Corporation for the U. S. Atomic Energy Commission.

[1] The low-frequency waves discussed in this paper are usually designated as magnetodynamic, magnetohydrodynamic, and hydromagnetic. Following the terminology of J. F. Denisse and J. L. Delcrox, *Theorie des Ondes dans les Plasmas* (Dunod Cie., Paris, 1961), the latter two designations are not used in this paper since the prefix "hydro" is associated with liquids and not with plasmas.

[2] Jacob Neufeld and Harvel Wright, Phys. Rev. **129**, 1489 (1963).

[3] For a definition of "dense plasma," see J. F. Denisse and J. L. Delcroix, *Theorie des Ondes dans les Plasmas* (Dunod Cie., Paris, 1961). The quantity "A" used by Denisse and Delcroix is approximately the quantity "A^2" as defined in (2.3).

[4] J. Dawson and I. B. Bernstein, paper presented at the Controlled Thermonuclear Conference, Washington, D. C., TID-7558, 360, 1958 (unpublished). I. B. Bernstein and K. Trehan, Nucl. Fusion **1**, 3 (1960).

[5] V. P. Dokuchaev, Zh. Eksperim. i Teor. Fiz. **39**, 413 (1960) [translation: Soviet Phys.—JETP **12**, 292 (1961)].

[6] K. N. Stepanov and A. B. Kitzenko, Zh. Tekhn. Fiz. **33**, 167 (1961) [translation: Soviet Phys.—Tech. Physics **6**, 120 (1962)].

[7] M. S. Kovner, Zh. Eksperim. i Teor. Fiz. **40**, 527 (1961) [translation: Soviet Phys.—JETP **13**, 369 (1961)].

[8] A. I. Akhiezer, A. B. Kitzenko, and K. N. Stepanov, Zh. Eksperim. i Teor. Fiz. **40**, 1866 (1961) [translation: Soviet Phys.—JETP **13**, 1311 (1961)].

An analysis is made of the two linearly polarized Alfvén waves (ordinary and extraordinary) excited by the beam and propagated at any angle θ with respect to the direction of the magnetic field $\mathbf{B}_0$. Expressions are derived for the frequency and rate of growth of these waves as a function of the velocity $c\beta$ of the beam (where c is the velocity of light), of the angle θ, and of the parameters of the stationary plasma. For $\theta=0$ and $\theta=\pi$ the magnetodynamic waves may be circularly polarized or linearly polarized. The instability behavior is different for these two types of polarization and some of these differences are described. A comparison is made between the velocities of electron and proton beams which are necessary in order to excite a magnetodynamic wave of a given frequency.

2. FORMULATION OF THE PROBLEM

The dispersion equation for a plasma-beam system can be expressed in a form[9]

$$Pn_r^4+Qn_r^2+R=0, \tag{2.1}$$

where n_r is the refractive index for a wave having frequency ω and wave number k. Both ω and k are positive. The quantities P, Q, and R are expressed in terms of various components of a tensor ϵ_{ij} which represents the modified capacitivity[10] of the plasma-beam system. The tensor ϵ_{ij} may be expressed as

$$\epsilon_{ij}=1+4\pi[(\chi_p^{(e)})_{ij}+(\chi_b^{(e)})_{ij}], \tag{2.2}$$

where $(\chi_p^{(e)})_{ij}$ is the electric susceptibility of the stationary plasma and $(\chi_b^{(e)})_{ij}$ is the modified electric susceptibility of the beam.

It is assumed that the plasma is comprised of electrons, and singly charged ions have mass M. Let M_p designate the mass of a proton in the incident beam. One has

$$\Omega_i=\frac{|e|B_0}{Mc} \quad\text{and}\quad \Omega_p=\frac{|e|B_0}{M_pc}, \tag{2.3}$$

where Ω_i is the ion gyrofrequency in the stationary plasma, Ω_p is the proton gyrofrequency for an observer moving with the beam, and $|e|$ is the electronic charge. Let

$$\omega_i=\left(\frac{4\pi ne^2}{M}\right)^{1/2} \quad\text{and}\quad \omega_p=\left(\frac{4\pi ne^2}{M_p}\right)^{1/2}, \tag{2.4}$$

where n is the density of electrons and singly charged ions in the plasma.

A very useful index for classifying various types of plasma is based on a parameter A^2 which is defined as

$$A^2=\omega_i^2/\Omega_i^2. \tag{2.5}$$

[9] E. Aström, Arkiv Fysik **2**, 443 (1950); A. G. Sitenko and K. N. Stepanov, Zh. Eksperim. i Teor. Fiz. **31**, 642 (1956) [translation: Soviet Phys.—JETP 4, 512 (1957)]; W. P. Allis, MIT Research Laboratory Electronics Quarterly Progress Report 54: 5, 1959 (unpublished).

[10] For a definition of "modified capacitivity," see Jacob Neufeld, Phys. Rev. **123**, 1 (1961).

According to the classification of Denisse and Delcroix[3] one has $A^2\sim 2.2\times 10^4$ for the ionosphere, $A^2\sim 1.4\times 10^2$ to 1.4×10^6 for solar corona, and $A^2\sim 4.3\times 10^{11}$ for interstellar clouds. All of these plasmas are characterized by $A^2\gg 1$. In order to obtain a magnetodynamic mode, one needs to have $A^2\gg 1$ and $\omega\ll\Omega_i$. Following the procedure outlined by Stepanov and Kitzenko, the relationship (2.1) is formulated in a rectangular x,y,z coordinate system in which the z axis (labeled as "3") is oriented in the direction $\mathbf{B}_0$, and the wave vector $\mathbf{k}$ is parallel to the x-z plane (labeled as 1-3 plane). Taking into account $A\gg 1$ and $\omega\ll\Omega_i$, the dispersion equations for a plasma-beam system as formulated by Stepanov and Kitzenko are as follows:

$$\frac{c^2k^2}{\omega^2}\cos^2\theta-\epsilon_{11}=0, \tag{2.6}$$

$$\frac{c^2k^2}{\omega^2}-\epsilon_{11}=0, \tag{2.7}$$

where ϵ_{11} is the 1-1 term of the matrix representing the modified capacitivity of the plasma-beam system.

In the absence of a beam, i.e., for $(\chi_b^e)_{11}=0$, Eq. (2.6) represents the ordinary Alfvén wave and (2.7) represents the extraordinary Alfvén wave.[11] These two waves will be labeled, respectively, as "Alfvén-O" and "Alfvén-X." The velocity distribution for an Alfvén-X wave is isotropic, i.e., the phase velocity ω/k of such a wave is independent of θ. Thus,

$$\omega/k=V_A=c/A=(B_0^2/4\pi nM)^{1/2}, \tag{2.8}$$

where V_A is the Alfvén velocity. On the other hand, the velocity distribution for an Alfvén-O wave is nonisotropic, i.e.,

$$\omega/k=V_A\cos\theta. \tag{2.9}$$

The dispersion equation for a charge-equilibrated system comprising a beam of protons passing through a stationary cold plasma is described by (2.6) and (2.7) in which

$$\epsilon_{11}=1-\frac{\omega_e^2}{\omega^2-\Omega_e^2}-\frac{\omega_i^2}{\omega^2-\Omega_i^2} -\frac{\sigma\omega_p^2(1-\beta^2)^{1/2}(\omega-ck\beta\cos\theta)^2}{\omega^2[(\omega-ck\beta\cos\theta)^2-\Omega_p^2(1-\beta^2)]}. \tag{2.10}$$

It is assumed that the density of the ion beam is σn (where $\sigma\ll 1$). The quantity β is positive when the beam

[11] The waves represented by (2.6) and (2.7) have been respectively labeled by E. Aström [Arkiv Fysik **2**, 443 (1950)] as extraordinary and ordinary. This designation has been reversed by W. P. Allis [MIT Research Laboratory Electronics Quarterly Progress Report 54: 5, 1959 (unpublished)]. In the terminology of Allis the wave represented by (2.6) is "ordinary" because for $\theta=\pi/2$ it is not affected by the magnetic field. The terminology used in this paper follows that of Allis.

moves in the direction of $\mathbf{B}_0$, and β is negative when the beam moves in the opposite direction.

In the expression (2.10),

$$\Omega_e = |e| B_0/mc \tag{2.11}$$

is the electron gyrofrequency, and

$$\omega_e = (4\pi ne^2/m)^{1/2} \tag{2.12}$$

is the plasma frequency (m is the mass of an electron).

The expression (2.10) represents a relativistic form of a corresponding expression obtained by Stepanov and Kitzenko.

3. EXCITATION OF ALFVÉN WAVE BY A PROTON BEAM

A. General Discussion

A. Dispersion Equations

Substituting (2.10) in (2.6) and (2.7), taking into account the magnetodynamic approximation ($A \gg 1$, $\omega \ll \Omega_i$), and assuming $\sigma \ll 1$, a simplified version of the dispersion equation is obtained. This equation is then as follows:

$$F(k,\theta,\omega) - \frac{\sigma\omega_p^2(1-\beta^2)^{1/2}(\omega - ck\beta\cos\theta)^2}{[(\omega - ck\beta\cos\theta)^2 - \Omega_p^2(1-\beta^2)]} = 0, \tag{3.1}$$

where $F(k,\theta,\omega)$ may be expressed in one of the following two forms:

$$F(k,\theta,\omega) = A^2\omega^2 - c^2k^2\cos^2\theta, \tag{3.2}$$

or

$$F(k,\theta,\omega) = A^2\omega^2 - c^2k^2. \tag{3.3}$$

The expression (3.2) represents an Alfvén-O wave and (3.3) represents an Alfvén-X wave.

Following the customary procedure,[2] Eq. (3.1) is solved for ω assuming that k is real. The roots of (3.1) are represented in the form

$$\omega = \tilde{\omega} + \delta, \tag{3.4}$$

where $\tilde{\omega}$ is the characteristic frequency and $\lim_{\sigma\to 0}\delta = 0$. When $\operatorname{Im}\delta < 0$ there is an instability, and the rate of growth of the excited wave is given by $|\operatorname{Im}(\delta)|$.

B. Resonant Beams

A beam of protons can be resonant or nonresonant with respect to a wave accompanying such a beam.[2] A resonance is said to occur when the frequency of the wave is equal to the gyrofrequency of protons in the beam. The frequency $\tilde{\omega}$ of the wave is measured in the laboratory system, while the measurement of the proton gyrofrequency Ω_p is made in the framework moving with the beam. The equality between these two frequencies, expressed by means of formulas (3.5) and (3.6), is formulated as follows:

$$\tilde{\omega} = ck\beta\cos\theta - \Omega_p(1-\beta^2)^{1/2}, \tag{3.5}$$

and

$$\tilde{\omega} = ck\beta\cos\theta + \Omega_p(1-\beta^2)^{1/2}. \tag{3.6}$$

Consider the expression (3.5). Since $\tilde{\omega} > 0$ and $\Omega_p > 0$, this implies $ck\beta\cos\theta > 0$. Hence, the following relationships hold:

$$\text{if } \beta > 0, \quad \text{then} \quad 0 \leqslant \theta < \tfrac{1}{2}\pi \tag{3.7}$$
$$\text{if } \beta < 0, \quad \text{then} \quad \tfrac{1}{2}\pi < \theta \leqslant \pi$$

and

$$ck\beta\cos\theta > \omega/k. \tag{3.8}$$

The expressions (3.7) are equivalent to the single inequality

$$\psi < \tfrac{1}{2}\pi, \tag{3.9}$$

where ψ is the angle formed by the direction of the beam and the direction of propagation of the wave.

The inequality (3.8) indicates that the velocity component of the beam in the direction of the wave exceeds the phase velocity of the wave. When the wave is aligned in the direction of the beam ($\psi = 0$), this inequality has the form

$$c\beta > \omega/k. \tag{3.10}$$

In accordance with the nomenclature used in Paper 1, the inequality (3.10) for $\psi = 0$ characterizes a superluminous beam. The term "superluminous" is now generalized for $\psi \neq 0$ and applies to a velocity satisfying the relationships (3.7) [or (3.9)] and (3.8). A resonant beam described by the equality (3.6) is seen not to be superluminous.

A beam may be superluminous but not resonant. In such a case the inequalities (3.7) and (3.8) are satisfied, but the equality (3.5) does not hold true.

Consider now the function $F(k,\theta,\omega)$ characterizing the two Alfvén modes in a stationary plasma. This function is expanded in a Taylor series about $\omega = \tilde{\omega}$, where $\tilde{\omega}$ may be expressed either as (3.5) or as (3.6). Retaining the first two terms of the series, one has

$$F(k,\theta,\omega) = (F)_{\tilde{\omega}} + \left(\frac{\partial F}{\partial \omega}\right)_{\tilde{\omega}} \delta, \tag{3.11}$$

where

$$(F)_{\tilde{\omega}} = [F(k,\theta,\omega)]_{\omega=\tilde{\omega}}; \quad \left(\frac{\partial F}{\partial \omega}\right)_{\tilde{\omega}} = \left[\frac{\partial F(k,\theta,\omega)}{\partial \omega}\right]_{\omega=\tilde{\omega}}. \tag{3.12}$$

It is assumed that for sufficiently small σ

$$|\delta| \ll \tilde{\omega}; \quad |\delta| \ll \Omega_p(1-\beta^2)^{1/2}. \tag{3.13}$$

Taking into account (3.4), (3.11), and (3.13), the dispersion equation (3.1) can be represented as

$$\delta^2\left(\frac{\partial F}{\partial \omega}\right)_{\tilde{\omega}} + \delta(F)_{\tilde{\omega}} \mp \frac{\sigma\omega_p^2(1-\beta^2)\Omega_p}{2} = 0. \tag{3.14}$$

The above equation describes an Alfvén-O wave if $F(k,\theta,\omega)$ is given by (3.2) and an Alfvén-X wave if $F(k,\theta,\omega)$ is given by (3.3). For a resonant, superluminous beam, i.e., when $\tilde{\omega}$ is expressed by (3.5), a positive sign should be applied to the term $\sigma\omega_p^2(1-\beta^2)\Omega_p/2$ in (3.14). When the resonant beam is not superluminous, i.e., for $\tilde{\omega}$ given by (3.6), a negative sign should be applied to this term.

The following relationship holds true for an Alfvén-O and an Alfvén-X wave:

$$\left(\frac{\partial F}{\partial \omega}\right)_{\varpi}=2A^2\tilde{\omega}. \tag{3.15}$$

Taking into account that $\tilde{\omega}>0$, $\Omega_i>0$, and (3.15), it can be shown that the discriminant of (3.14) is always positive when the negative sign is applied to the term $\sigma\omega_p^2(1-\beta^2)\Omega_p/2$. Hence, the roots of (3.14) are real, implying that there is no instability when the beam is resonant but not superluminous.

For a resonant superluminous beam the roots of (3.14) are complex if

$$(F)_{\varpi}^2<2\sigma\left(\frac{\partial F}{\partial \omega}\right)_{\varpi}\omega_p^2(1-\beta^2)\Omega_p. \tag{3.16}$$

Since $\sigma\rightarrow 0$, the inequality (3.16) becomes equivalent to

$$(F)_{\varpi}=0. \tag{3.17}$$

For a wave resonant with a superluminous beam, one has

$$\omega=\tilde{\omega};\quad k=\frac{\tilde{\omega}+\Omega_p(1-\beta^2)^{1/2}}{c\beta\cos\theta}. \tag{3.18}$$

Consequently, the characteristic frequency of an excited Alfvén wave is obtained from the equation

$$\phi(\beta,\tilde{\omega},\theta)=0, \tag{3.19}$$

where the expression $\phi(\beta,\tilde{\omega},\theta)$ is obtained by substituting in the expression $F(k,\theta,\omega)$ the values ω and k given in (3.18).

The rate of growth of an Alfvén wave excited by a superluminous beam is obtained from (3.14) and is as follows:

$$|\mathrm{Im}(\delta)|=\frac{\sigma^{1/2}\Omega_p^{3/2}(1-\beta^2)^{1/2}}{2\tilde{\omega}^{1/2}\alpha^{1/2}}, \tag{3.20}$$

where $\alpha=M_i/M_p$.

C. Nonresonant Beams

For a nonresonant beam, the equality between the frequency of the wave and the gyrofrequency of protons in the beam does not exist. Therefore, the frequency of a nonresonant wave does not satisfy the relationship (3.5) or (3.6). The nonresonant waves are of no particular interest in this investigation since these waves cannot be excited by a beam. This was shown in Paper 1 for the case of $\psi=0$. A similar procedure can be applied in order to show that there is no instability for nonresonant waves when $\psi\neq 0$.

B. Alfvén-*O* Mode

The excited frequency $\tilde{\omega}$ and the rate of growth $|\mathrm{Im}(\delta)|$ will now be determined for Alfvén-O and Alfvén-X waves. The quantities $\tilde{\omega}$ and $|\mathrm{Im}(\delta)|$ will be expressed either as a function of β, V_A, Ω_i, θ, α, or as a function of β, A, Ω_i, θ, α.

The characteristic frequency $\tilde{\omega}$ of an excited Alfvén-O wave is obtained from the equality (3.19). One has

$$\tilde{\omega}=\frac{\Omega_p(1-\beta^2)^{1/2}V_A}{c\beta-V_A}=\frac{\Omega_p(1-\beta^2)^{1/2}}{A\beta-1}. \tag{3.21}$$

Expression (3.21) is subject to the inequality $\tilde{\omega}\ll\Omega_p$. Therefore, the following inequalities have to be satisfied:

$$\alpha(1-\beta^2)\ll(c\beta/V_A)-1,\ \text{or}\ \alpha(1-\beta^2)^{1/2}\ll A\beta-1. \tag{3.22}$$

The rate of growth $|\mathrm{Im}(\delta)|$ for an Alfvén-O mode is obtained from (3.20) and is as follows:

$$|\mathrm{Im}(\delta)|=\frac{\sigma^{1/2}\Omega_p(1-\beta^2)^{1/4}(c\beta-V_A)^{1/2}}{2V_A^{1/2}\alpha^{1/2}}=\frac{\sigma^{1/2}(1-\beta^2)^{1/4}\Omega_p(A\beta-1)^{1/2}}{2\alpha^{1/2}}. \tag{3.23}$$

Consider now the nondimensional parameter $N=|\mathrm{Im}(\delta)|/\tilde{\omega}$ representing the relative rate of growth (in decibels per cycle) of an Alfvén-O wave. This quantity can be expressed as follows:

$$N=\frac{\sigma^{1/2}(c\beta-V_A)^{3/2}}{2(1-\beta^2)^{1/4}V_A^{3/2}\alpha^{1/2}}=\frac{\sigma^{1/2}(A\beta-1)^{3/2}}{2(1-\beta^2)^{1/4}\alpha^{1/2}}. \tag{3.24}$$

The expressions (3.23) and (3.24) are subject to the inequalities (3.22).

It is of interest to note the angular behavior of the Alfvén-O mode. This mode has a nonisotropic velocity distribution given by (2.9). However, both the excited frequency $\tilde{\omega}$ and the rate of growth $|\mathrm{Im}(\delta)|$ are "isotropic," i.e., these two quantities have the same value for any angle θ.

For a nonrelativistic beam ($\beta\ll 1$), the two inequalities (3.22) may be expressed as follows:

$$c\beta/V_A\gg\alpha;\quad \beta A\gg\alpha. \tag{3.25}$$

One of the inequalities (3.25) states that the beam velocity must be very large when compared to the Alfvén velocity. This requirement is more stringent than the one expressed by (3.22). In accordance with the other inequality (3.25), a very dense plasma is necessary ($A\gg 1$) in order to excite an Alfvén wave by a nonrelativistic beam.

The characteristic frequency $\tilde{\omega}$ excited by a nonrelativistic beam is as follows:

$$\tilde{\omega}=\Omega_pV_A/c\beta=\Omega_p/A\beta, \tag{3.26}$$

and the rate of growth is

$$|\mathrm{Im}(\delta)|=\frac{\sigma^{1/2}\Omega_p(c\beta)^{1/2}}{2V_A^{1/2}\alpha^{1/2}}=\frac{\sigma^{1/2}\Omega_pA^{1/2}\beta^{1/2}}{2\alpha^{1/2}}. \qquad (3.27)$$

The expressions (3.26) and (3.27) are subject to the inequality (3.25).

C. Alfvén-X Mode

The characteristic frequency $\tilde{\omega}$, the rate of growth $|\mathrm{Im}(\delta)|$, and the relative rate of growth N of an excited Alfvén-X wave are as follows:

$$\tilde{\omega}=\frac{\Omega_p(1-\beta^2)^{1/2}V_A}{c\beta\cos\theta-V_A}=\frac{\Omega_p(1-\beta^2)^{1/2}}{A\beta\cos\theta-1}, \qquad (3.28)$$

$$|\mathrm{Im}(\delta)|=\frac{\sigma^{1/2}\Omega_p(1-\beta^2)^{1/4}(c\beta\cos\theta-V_A)^{1/2}}{2V_A^{1/2}\alpha^{1/2}}$$

$$=\frac{\sigma^{1/2}\Omega_p(1-\beta^2)^{1/4}(A\beta\cos\theta-1)^{1/2}}{2\alpha^{1/2}}, \qquad (3.29)$$

$$N=\frac{\sigma^{1/2}(c\beta\cos\theta-V_A)^{3/2}}{2(1-\beta^2)^{1/4}V_A^{3/2}\alpha^{1/2}}$$

$$=\frac{\sigma^{1/2}(A\beta\cos\theta-1)^{3/2}}{2(1-\beta^2)^{1/4}\alpha^{1/2}}. \qquad (3.30)$$

The expression (3.28) is subject to the inequality $\tilde{\omega}\ll\Omega_i$. Therefore, combining (3.28) with $\tilde{\omega}\ll\Omega_i$, one obtains an inequality which can be expressed in one of the following two forms:

$$\alpha(1-\beta^2)^{1/2}\ll\frac{c\beta\cos\theta}{V_A}-1$$

or

$$\alpha(1-\beta^2)^{1/2}\ll A\beta\cos\theta-1. \qquad (3.31)$$

When θ is small, the inequalities (3.31) differ very little from the inequalities (3.22) which are applicable to the Alfvén-O mode. Larger values of θ require the product $A\beta$ to be correspondingly larger. In the neighborhood of $\theta=\frac{1}{2}\pi$ the inequalities (3.31) require the product $A\beta$ to be extremely large. The expressions (3.28) to (3.30) are subject to inequalities (3.31).

For a nonrelativistic beam, the above expressions become

$$\tilde{\omega}=\frac{\Omega_pV_A}{c\beta\cos\theta}=\frac{\Omega_p}{A\beta\cos\theta}, \qquad (3.32)$$

$$|\mathrm{Im}(\delta)|=\frac{\sigma^{1/2}\Omega_p(c\beta\cos\theta)^{1/2}}{2V_A^{1/2}\alpha^{1/2}}=\frac{\sigma^{1/2}\Omega_p(A\beta\cos\theta)^{1/2}}{2\alpha^{1/2}}. \qquad (3.33)$$

The expressions (3.32) and (3.33) are subject to the inequality

$$\frac{c\beta\cos\theta}{V_A}\gg\alpha,\quad\text{or}\quad \beta A\cos\theta\gg\alpha. \qquad (3.34)$$

The angular behavior of the Alfvén-X wave is different from that of the Alfvén-O wave discussed above. For an Alfvén-X wave the velocity distribution is isotropic since $(\omega/k)=V_A$ for all values of θ. However, both the excited frequency $\tilde{\omega}$ and the rate of growth $|\mathrm{Im}(\delta)|$ are anisotropic since these quantities vary with θ.

4. EXCITATION OF AN ALFVÉN WAVE BY AN ELECTRON BEAM

The dispersion equation for a charge-equilibrated system comprising an electron beam having density σn passing through a stationary plasma is described by (2.6) and (2.7), in which

$$\epsilon_{11}=1-\frac{\omega_e^2}{\omega^2-\Omega_e^2}-\frac{\omega_i^2}{\omega^2-\Omega_i^2}-\frac{\sigma\omega_e^2(1-\beta^2)^{1/2}(\omega-ck\beta\cos\theta)^2}{\omega^2[(\omega-ck\beta\cos\theta)^2-\Omega_e^2(1-\beta^2)]}. \qquad (4.1)$$

Consequently, in order to obtain the dispersion equation for an electron beam, one needs to replace in the dispersion equation for a proton beam ω_p by ω_e and Ω_p by Ω_e. Thus, the various expressions for the excited frequency, rate of growth, etc., for an electron beam can be obtained from the corresponding expressions for an ion beam by replacing in various formulas as (3.21), (3.23), (3.24), and (3.26) to (3.30) the quantities ω_p and Ω_p by ω_e and Ω_e, respectively.

In order to excite a wave, an electron beam has to be resonant and superluminous. The resonance condition is expressed by (3.5) and (3.6) in which Ω_p has to be replaced by Ω_e. The electron beam is superluminous if it satisfies the same conditions, (3.7) [or (3.9)] and (3.8), as a superluminous ion beam.

In order to excite an Alfvén-O wave, the velocity β of an electron beam has to satisfy the following inequalities:

$$\eta(1-\beta^2)^{1/2}\ll\frac{c\beta}{V_A}-1,\quad\text{or}\quad \eta(1-\beta^2)^{1/2}\ll A\beta-1, \qquad (4.2)$$

where

$$\eta=M/m. \qquad (4.3)$$

Similarly, for the excitation of an Alfvén-X wave, the following inequality has to be satisfied:

$$\eta(1-\beta^2)^{1/2}\ll(c\beta\cos\theta/V_A)-1,$$

or

$$\eta(1-\beta^2)^{1/2}\ll A\beta\cos\theta-1. \qquad (4.4)$$

5. EXCITATION OF A MAGNETODYNAMIC WAVE ALIGNED IN THE DIRECTION OF THE BEAM

A. General Discussion

Consider a resonant superluminous ion beam aligned along the direction of the magnetic field, i.e., assume that $\theta=0$ or $\theta=\pi$. In such a case, the magnetodynamic waves may be linearly or circularly polarized. A brief discussion will be given on the excitation of the circularly polarized waves.

The dispersion equation for circularly polarized waves propagating along the magnetic field $\mathbf{B}_0$ and excited by an incident proton beam have been expressed in Paper 1 in the following form:

$$F(k,\omega)-\frac{\sigma(1-\beta^2)^{1/2}\omega_p{}^2(\omega-ck\beta)}{\omega-ck\beta+\Omega_p(1-\beta^2)^{1/2}}=0, \tag{5.1}$$

where, taking into account the inequality $\sigma\ll1$,

$$F(k,\omega)=\omega^2-c^2k^2-\frac{\omega_i{}^2\omega}{\omega+\Omega_i}-\frac{\omega_e{}^2\omega}{\omega-\Omega_e}. \tag{5.2}$$

The symbols used in (5.1) and (5.2) have the same meaning as those in (3.1), except for the symbol labeled "ω." In (3.1), ω designates the frequency of a linearly polarized wave and is always positive. On the other hand, in (5.1) ω represents an angular frequency of a circularly polarized wave and it may be positive or negative. Following the notation of Paper 1, $\omega>0$ (or $\omega<0$) indicates in (5.1) clockwise (or anticlockwise) rotation when the observer is looking in the positive direction. The sign of k determines the helicity of the wave. Thus, $k>0$ indicates an H_+ wave and $k<0$ indicates an H_- wave.

Using the magnetodynamic approximation ($A\gg1$; $\omega\ll\Omega_i$), Eq. (5.1) can be modified, and applying the formulation of Paper 1, one can determine various parameters characterizing excited H_+ and H_- waves. The results will be summarized considering, in particular, the two cases of a beam moving (1) in the direction of and (2) opposite to the direction of the magnetic field.

B. A Proton Beam Moving in the Direction of the Magnetic Field ($\beta>0$)

In this case the beam and the magnetodynamic wave move in the direction of the magnetic field. Only H_+ waves can be excited and H_- waves remain stationary. The characteristic angular frequency $\tilde{\omega}$ of the excited H_+ wave and its rate of growth can be expressed as follows:

$$\tilde{\omega}=\frac{\Omega_p(1-\beta^2)^{1/2}V_A}{c\beta-V_A}=\Omega_p(1-\beta^2)^{1/2}\frac{S}{1-S} \tag{5.3}$$

and

$$|\mathrm{Im}(\delta)|=\frac{1}{\sqrt{2}}\frac{\sigma^{1/2}\Omega_p(1-\beta^2)^{1/4}(c\beta-V_A)^{1/2}}{(c\beta)^{1/2}\alpha^{1/2}}=\frac{\sigma^{1/2}\Omega_p(1-\beta^2)^{1/4}(1-S)^{1/2}}{(2S)^{1/2}\alpha^{1/2}}. \tag{5.4}$$

The term S in the above expression represents a velocity index which is the ratio of the Alfvén velocity to the velocity of the beam, i.e.,

$$S=V_A/c\beta. \tag{5.5}$$

C. A Proton Beam Moving in the Direction Opposite to that of the Magnetic Field ($\beta<0$)

A proton beam moving in the direction opposite to that of $\mathbf{B}_0$ can excite only H_- waves moving with the beam, whereas H_+ waves remain stationary. The characteristic angular frequency of the excited H_- wave is given by (5.3), and the corresponding rate of growth is expressed by (5.4).

D. Instability of a Linearly Polarized Alfvén Wave

A linearly polarized Alfvén wave can be represented by a superposition of an H_+ and an H_- wave propagating in the same direction. In order to analyze the structural behavior of a linearly polarized Alfvén wave, one should differentiate between the wave excited by a beam moving in the direction of $\mathbf{B}_0$ and opposite to the direction of $\mathbf{B}_0$. If a proton beam moves in the direction of the magnetic field, then only the H_+ component of an Alfvén wave is excited, and the H_- component remains stationary. This relationship is reversed for a proton beam moving against the magnetic field.

6. GENERAL COMMENTS

Dokuchaev,[5] Stepanov and Kitzenko,[6] and Kovner[7] considered an instability in which a beam having velocity V_B and density n' would excite magnetodynamic waves in a cold plasma. Their formulation of the problem was different from that of this investigation and the criterion for instability was expressed there as follows:

$$V_B{}^2>V_A{}^2+V_A{}'^2, \tag{6.1}$$

where

$$V_A=\left(\frac{B_0{}^2}{4\pi nM}\right)^{1/2};\quad V_A{}'=\left(\frac{B_0{}^2}{4\pi n'M}\right)^{1/2}. \tag{6.2}$$

It appears from (6.1) that for a limiting case of $n'\to0$ the magnetodynamic excitation would require a beam of infinite velocity. Such a requirement has not been obtained in this investigation. Compare in that connection the inequality (6.1) with the corresponding inequalities (3.22), (3.25), (3.31), (3.34), (4.2), and (4.4). For instance, the inequality (3.25) states that a nonrelativistic beam may excite an Alfvén-O wave when the velocity of the beam is very large when compared to the Alfvén velocity. However, this velocity need not be infinite, as indicated by (6.1). Furthermore, for the beams of small density ($\sigma \ll 1$), the restrictions imposed on the velocity of the beam are independent of the quantity σ. On the other hand, the inequality (6.1) is very sensitive to the value representing the density of the beam.

In some of the above investigations, the inadequacy of the formulation (6.1) was recognized and certain additional restrictive conditions were postulated. Thus, according to Stepanov and Kitzenko,[6] one may obtain valid results for $n' \rightarrow 0$, if thermal motion of the incident beam is taken into account. Similarly, according to Akhiezer, Kitzenko, and Stepanov,[8] the coupling between the Alfvén wave and the magnetoacoustic wave should be taken into account. It is believed, however, that these restrictive conditions are not necessary in order to obtain valid results.

The discrepancy between the results of this investigation and those represented by (6.1) is based on the initial assumption concerning the character of the magnetodynamic waves which can be excited by an incident beam. As has been pointed out in this investigation, the excited magnetodynamic waves must be resonant with the beam and the waves which are not resonant with the beam cannot be excited by a beam of sufficiently small density.

It is to be noted, however, that the above resonance condition was not assumed in the calculating leading to (6.1). Thus, Dokuchaev[5] and Akhiezer, Kitzenko, and Stepanov[8] assumed that the excited Alfvén wave has to satisfy the inequality

$$ck\beta \ll \Omega_p, \tag{6.3}$$

whereas Stepanov and Kitzenko[6] and Kovner[7] assumed that

$$|\omega - ck\beta| \ll \Omega_p, \quad \text{or} \quad \omega - ck\beta \ll \Omega_p, \tag{6.4}$$

where ω represents the frequency of an Alfvén wave. The inequalities (6.3) and (6.4) are not compatible with an assumption that an Alfvén wave must be resonant with a beam of charged particles comprising ions or electrons. Using the nonrelativistic case for $\theta = 0$ and assuming linearly polarized waves ($\omega > 0$), the resonance condition between a proton beam and an electromagnetic wave can be expressed as

$$\omega - ck\beta = -\Omega_p. \tag{6.5}$$

The corresponding resonance condition for an electron beam is as follows:

$$\omega - ck\beta = -\Omega_e. \tag{6.6}$$

Neither of the inequalities (6.3) or (6.4) is satisfied if one assumes that the relationship (6.5) or (6.6) is valid.